The Scots College

2008
TRIAL HSC EXAMINATION

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using a blue or black pen
- Board approved calculators may be used
- A standard table of integrals is provided
- All necessary working should be shown in every question

Total Marks: 84
Weighting: 40% HSC

- Attempt Questions 1 – 7
- All questions are of equal value

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Question 1 (Start a new booklet)

(a) Determine the coordinates of the point P that divides the interval joining $A(-1 , 6)$ and $B(4 , -6)$ externally in the ratio 2:3. **[2 marks]**

(b) Solve for x: $\frac{x}{2x-1} \le -2$ **[3 marks]**

(c) Give a general solution to the equation $\cos x + \frac{1}{2} = 0$.
Leave your answer in terms of π. **[3 marks]**

(d) Use the substitution $u = \sin x$ to evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cot x \, dx$. **[4 marks]**

Question 2 (Start a new booklet)

(a) **(i)** State the domain and range of the function $y = 3\cos^{-1} 2x$. **[2 marks]**

(ii) Sketch the graph of $y = 3\cos^{-1} 2x$, clearly showing the intercepts on the axes and the coordinates of any endpoints. **[1 mark]**

(iii) Find the area of the region bounded by the curve $y = 3\cos^{-1} 2x$, the coordinate axes, in the first quadrant. **[2 marks]**

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Question 2 continued

(b) **(i)** Express $\cos x + \sqrt{3}\sin x$ in the form $R\sin(x+\alpha)$ for $R > 0$. **[2 marks]**

(ii) Hence or otherwise state the least value of $\cos x + \sqrt{3}\sin x$. **[1 mark]**

(c) The volume of a cube is increasing at the rate of $30\,cm^3/\min$.
Let $x\,cm$ represent the edge length of the cube.

(i) Show that $\frac{dx}{dt} = \frac{10}{x^2}\ cm/\min$. **[2 marks]**

(ii) Hence find the rate at which the surface area of the cube is increasing when the edge length is $10\,cm$. **[2 marks]**

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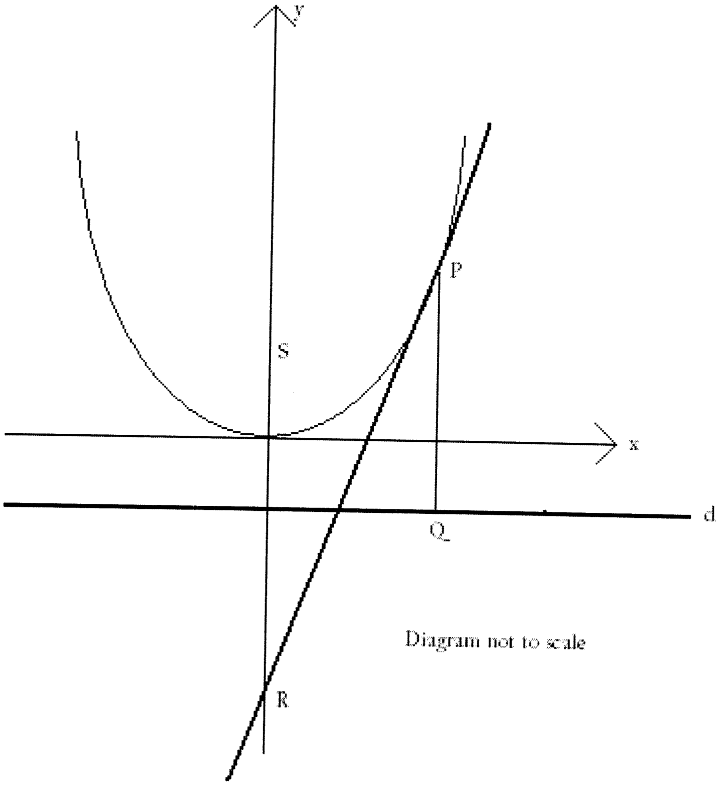
Question 3 (Start a new booklet)

(a) Find $\int \cos^2 4x \, dx$. **[2 marks]**

(b) In the expansion $(1-2x)(1+ax)^{10}$ the coefficient of x^6 is zero. Find the value of a. **[3 marks]**

(c)



$P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$. S is the focus of the parabola. PQ is the perpendicular from P to the directrix d of the parabola. The tangent at P to the parabola cuts the axis of the parabola at the point R.

(i) Show that the tangent at P to the parabola has equation $tx - y - at^2 = 0$. **[2 marks]**

(ii) Show that PR and QS bisect eachother. **[2 marks]**

(iii) Show that PR and QS are perpendicular to eachother. **[2 marks]**

(iv) What type of quadrilateral is $PQRS$? Giving reasons. **[1 mark]**

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Question 4 (Start a new booklet)

(a) Prove by mathematical induction that, for all positive integers n, **[4 marks]**

$$\sum_{r=1}^{n}(r^2+1)r! = n(n+1)!$$

(b) The rate of decrease of temperature of a body hotter than its surroundings is proportional to the temperature difference.

If A is the air temperature and T is the temperature of the body after t minutes then $\frac{dT}{dt} = -k(T-A)$.

(i) Show that, if I is the initial temperature of the body, then $T = A+(I-A)e^{-kt}$ satisfies the above equation. **[1 mark]**

(ii) If the temperature of the body is $1400^\circ C$ and it cools in the open where the temperature is $20^\circ C$, find, to the nearest degree, the temperature after 30 minutes, given that it cooled to $1200^\circ C$ in 5 minutes. **[3 marks]**

(c) TA is a tangent which touches the circle at T. TA is extended to P.
C is a point on the circumference of this circle such that CA cuts the circle at B.
Also $CB = BT$.

(i) Draw a diagram using the above information. **[1 mark]**

(ii) Prove that $\angle BAP = 3\times\angle BTA$, giving reasons. **[3 marks]**

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Question 5 (Start a new booklet)

(a) If the roots of a cubic polynomial are 0, 1 and 3 and the coefficient of x^3 is 2, find the equation of the polynomial. **[2 marks]**

(b) **(i)** Show that $e^x + x = 3$ has a root between $x = 0$ and $x = 1$. **[1 mark]**

(ii) By taking $x = 0.8$ as an approximate solution, use one application of Newton's Method to find a better approximation, correct to 3 significant figures. **[2 marks]**

(c) A particle, moving on a straight line is x metres from an origin after t seconds, where $\frac{d^2x}{dt^2} = -4x$.

(i) Verify that the position function $x = A\cos(2t + \alpha)$, where A and α are constants, satisfies $\frac{d^2x}{dt^2} = -4x$. **[1 mark]**

(ii) Prove that its velocity, $v\ ms^{-1}$, is given by $v^2 = 4(A^2 - x^2)$. **[2 marks]**

(iii) If the particle is at rest at $x = 5$, find the value of A, given that it is positive. **[1 mark]**

(iv) Initially, the particle is at $x = 3$ with velocity $8\ ms^{-1}$. Find the values of both $\cos\alpha$ and $\sin\alpha$ and the exact position of the particle when $t = \frac{\pi}{6}$. **[3 marks]**

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Question 6 (Start a new booklet)

(a) Evaluate $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}}$. **[2 marks]**

(b) **(i)** Write down the Binomial expansion of $(1+x)^n$ in ascending powers of x. **[1 mark]**

(ii) Find the value of n such that the coefficient of x^4 is twice the coefficient of x^3. **[3 marks]**

(c) A missile is released from a plane flying horizontally at a height of 500 metres and at a constant speed of 300 metres per second.

(i) Show that the equation of the horizontal component of the subsequent motion of the missile is given by $x = 300t$, where t is time and x is the horizontal displacement of the missile. **[2 marks]**

(ii) A rocket launcher on the ground is immediately below the plane at the instant the missile is released. Two seconds later it is fired in an attempt to intercept the missile. It is fired in the same direction and in the same vertical plane as the missile. It is fired at an angle of $60°$ to the horizontal at a speed of 1000 metres per second. **[3 marks]**

Determine how long after firing a possible interception occurs.

(iii) Explain briefly how you would determine whether the missile is successfully intercepted. **[1 mark]**

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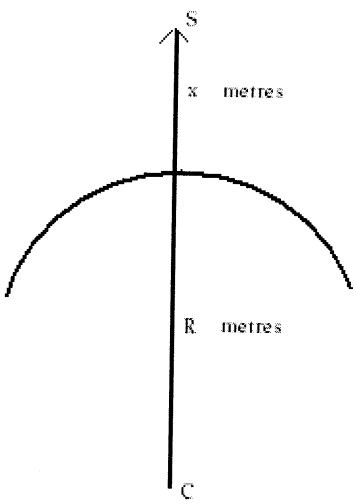
Question 7 (Start a new booklet)

(a) By considering the binomial expansion of $(1+x)^n$, and its derivative, prove that:

(i) $^nC_1 + 2\,^nC_2 + 3\,^nC_3 + \ldots + n\,^nC_n = n(2^{n-1})$ **[3 marks]**

(ii) $\sum_{r=0}^{n}(r+1)\,^nC_r = 2^{n-1}(2+n)$ **[3 marks]**

(b)



A space shuttle, *S*, is fired from the surface of the Earth, away from its centre, with speed $U\ ms^{-1}$. After travelling *x* metres, its acceleration is given by $\frac{-k}{(x+R)^2}\ ms^{-2}$, where *k* is constant, and *R* is the radius of the Earth.

(i) Prove that, at this point, its velocity, $v\ ms^{-1}$ is given by **[3 marks]**

$$v^2 = U^2 - \frac{2kx}{R(x+R)}$$

(ii) If the acceleration at the Earth's surface is denoted by $-g\ ms^{-2}$, prove that **[2 marks]**

$$v^2 = U^2 - \frac{2gRx}{x+R}$$

(iii) If *x* is small compared with *R*, so that $\frac{x}{R}$ is close to zero, **[1 mark]**

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show that the relation in (ii) may be written $v^2 = U^2 - 2gx$.

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Ext1 Maths Trial Solutions, August 2008

Question 1

(a)

$$x = \frac{nx_1 + mx_2}{m+n} \qquad y = \frac{ny_1 + my_2}{m+n}$$

$$x = \frac{3(-1) - 2(4)}{-2+3} \qquad y = \frac{3(6) - 2(-6)}{-2+3}$$

$$x = -11 \qquad y = 30$$

$\therefore P(-11, 30)$

(b)

$$\frac{x}{2x-1} \le -2$$
$$x(2x-1) \le -2(2x-1)^2$$
$$2(2x-1)^2 + x(2x-1) \le 0$$
$$(2x-1)[2(2x-1)+x] \le 0$$
$$(2x-1)(5x-2) \le 0$$
$$\therefore \frac{2}{5} \le x < \frac{1}{2}$$

(c)

$$\cos x + \frac{1}{2} = 0$$
$$\cos x = -\frac{1}{2}$$
$$x = 2n\pi \pm \frac{2\pi}{3}$$

(d)

$$u = \sin x$$
$$\frac{du}{dx} = \cos x$$

When $x = \frac{\pi}{6}, u = \frac{1}{2}$

When $x = \frac{\pi}{2}, u = 1$

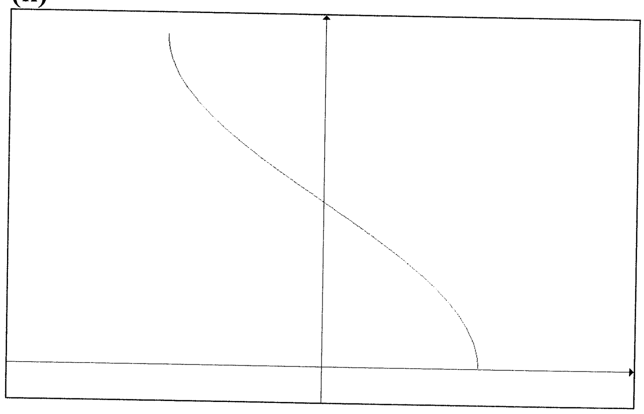
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cot x \, dx = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx$$
$$= 2\int_{\frac{1}{2}}^{1} \frac{du}{u}$$
$$= 2\left[\ln u\right]_{\frac{1}{2}}^{1}$$
$$= 2[\ln 1 - \ln \frac{1}{2}]$$
$$= 2\ln 2$$

Question 2

(a) $y = 3\cos^{-1} 2x$

(i) D: $-\frac{1}{2} \le x \le \frac{1}{2}$ R: $0 \le y \le 3\pi$

(ii)



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$$A=\frac{1}{2}\int_0^{\frac{3\pi}{2}}\cos\frac{y}{3}dy$$

(iii) $$A=\frac{1}{2}.3[\sin\frac{y}{3}]_0^{\frac{3\pi}{2}}$$

$$A=\frac{3}{2}[\sin\frac{\pi}{2}-\sin 0]$$

$$A=\frac{3}{2}\text{ sq units}$$

(b)

(i) $\cos x+\sqrt{3}\sin x = R\sin(x+\alpha)$

$\cos x+\sqrt{3}\sin x = R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha=\sqrt{3}$

$R\sin\alpha = 1$

$\therefore \tan\alpha=\frac{1}{\sqrt{3}}$

$\therefore \alpha=\frac{\pi}{6}$

$R^2=(\sqrt{3})^2+1^2$

$R^2=4$

$\therefore R=2$

since $R>0$

$\therefore \cos x+\sqrt{3}\sin x = 2\sin(x+\frac{\pi}{6})$

(ii) $\therefore$ least value of $\cos x+\sqrt{3}\sin x$ is -2

(c)

$V=x^3$

(i) $\therefore \frac{dV}{dx}=3x^2$

$\frac{dV}{dt}=30$

$\frac{dV}{dt}=\frac{dV}{dx}\times\frac{dx}{dt}$

$30=3x^2\times\frac{dx}{dt}$

$\therefore \frac{dx}{dt}=\frac{10}{x^2}$

$A=6x^2$

$\therefore \frac{dA}{dx}=12x$

(ii) $\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}$

$\frac{dA}{dt}=12x\times\frac{10}{x^2}$

At $x=10$

$\frac{dA}{dt}=12(10).\frac{10}{100}=12\,cm^2/\text{second}$

Question 3

(a)

$$\int\cos^2 4x\,dx=\frac{1}{2}\int\cos 8x+1dx$$
$$=\frac{1}{2}[\frac{\sin 8x}{8}+x]+c$$

(b)

$(1-2x)(1+ax)^{10}$

$=(1-2x)(^{10}C_0+^{10}C_x ax+...+^{10}C_5a^5x^5+^{10}C_6a^6x^6+...+^{1(}$

coefficient of $x^6=0$

ie $^{10}C_6a^6-2.^{10}C_5a^5=0$

$210a^6-504a^5=0$

$a^5(210a-504)=0$

$\therefore a=0\text{ or }a=\frac{12}{5}$

but $a\neq 0, \therefore a=\frac{12}{5}$

(c)

$P(2at,at^2)$, $S(a,0)$, $x^2=4ay$

(i)

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$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

At $x = 2at$, $\frac{dy}{dx} = t$

Eqn of tangent at P

$y - at^2 = t(x - 2at)$

$y - at^2 = tx - 2at^2$

$\therefore tx - y - at^2 = 0$

(ii)

sub $x = 0$, $y = at^2$

$\therefore P(2at, at^2)$, $R(0, -at^2)$

Midpoint of PR

$= \left(\frac{2at+0}{2}, \frac{at^2 - at^2}{2}\right)$

$= (at, 0)$

$S(0, a)$, $Q(2at, -a)$

Midpoint SQ

$= \left(\frac{2at+0}{2}, \frac{-a+a}{2}\right)$

$= (at, 0)$

Since the midpoint are the same

Then PR bisects SQ

(iii)

$m_{PR} = t$

$m_{SQ} = \frac{a+a}{0-2at}$

$= \frac{-1}{t}$

Since $m_{PR} \times m_{SQ} = -1$

$\therefore PR \perp SQ$

(iv)

$PQRS$ is a rhombus since the diagonal bisect each other at right angles.

Question 4

(a)

$$\sum_{r=1}^{n} (r^2+1)r! = n(n+1)!$$

ie $2.1!+5.2!+10.3!+\ldots+(n^2+1).n! = n(n+1)!$

Step 1: Need to prove that $n = 1$ is true

LHS $= 2 \times 1! = 2$

RHS $= 1 \times 2! = 2 =$ LHS

$\therefore n = 1$ is true

Step 2: Assume $n = k$ is true

ie $2.1!+5.2!+10.3!+\ldots+(k^2+1).k! = k(k+1)!$

Need to prove that $n = k + 1$ is also true

ie

$2.1!+5.2!+10.3!+\ldots+(k^2+1).k!+((k+1)^2+1)(k+1)!$
$= (k+1)(k+2)!$

LHS

$= 2.1!+5.2!+10.3!+\ldots+(k^2+1).k!+((k+1)^2+1)(k+1)!$

$= k(k+1)!+((k+1)^2+1)(k+1)!$

$= (k+1)!\left[k + k^2 + 2k + 1 + 1\right]$

$= (k+1)!(k^2+3k+2)$

$= (k+1)!(k+2)(k+1)$

$= (k+1)(k+2)!$

$= RHS$

$\therefore n = k+1$ is also true

Step 3:

Since $n = 1$, $n = k$ and $n = k + 1$ are all true

Then $n = 2$, $n = 3$, …are also true

$$\therefore \sum_{r=1}^{n} (r^2+1)r! = n(n+1)!$$

(b)

$T = A + (I - A)e^{-kt}$

(i)

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$$\frac{dT}{dt} = (I - A)e^{-kt}.-k$$
$$= -k(T - A)$$

(ii)

$I = 1400 \qquad A = 20$

$t = 5 \quad T = 1200$

$T = A + (I - A)e^{-kt}$

$\therefore\ T = 20 + (1400 - 20)e^{-kt}$

$\therefore\ T = 20 + 1380e^{-kt}$

sub $t = 5 \qquad T = 1200$

$\therefore\ 1200 = 20 + 1380e^{-5t}$

$e^{-5t} = \dfrac{1180}{1380}$

$-5k = \ln\left(\dfrac{1180}{1380}\right)$

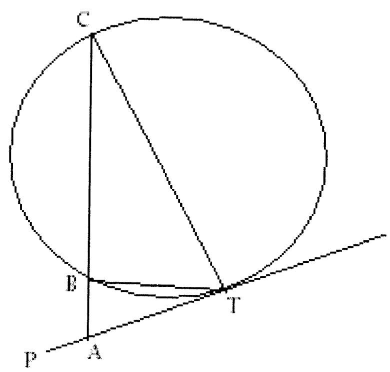
$\therefore k = 0.0313...$

sub $t = 30$, $\therefore\ T = 20 + 1380e^{-0.0313(30)}$

$\therefore\ T = 559^\circ C$ (nearest degree)

(c)

(i)



(ii)

Join CT

Let $\angle BTA = x$

$\angle BCT = x$ (angle in alt segment)

$\angle BCT = \angle BTC = x$ (base angles of iso triangle)

$\angle BAP = x + x + x$ (ext angle of triangle)

$\therefore \angle BAP = 3x = 3\angle BTA$

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Question 5

(a) roots 0, 1 and 3

$P(x) = 2x(x-1)(x-3)$

$P(x) = 2x(x^2 - 4x + 3)$

$P(x) = 2x^3 - 8x^2 + 6x$

(b)

$e^x + x = 3$

Let $f(x) = e^x + x - 3$

$f(0) = e^0 + 0 - 3 = -2$

$f(1) = e^1 + 1 - 3 = 0.718...$

Since $f(0)$ and $f(1)$ are opposite signs

$\therefore$ root between $x = 0$ and $x = 1$.

(ii)

$f(x) = e^x + x - 3$

$f'(x) = e^x + 1$

$x_1 = 0.8 - \dfrac{f(0.8)}{f'(0.8)}$

$x_1 = 0.8 - \dfrac{0.0255...}{3.2255...}$

$x_1 = 0.792$ (3 sig fig)

(c)

(i)

$x = A\cos(2t + \alpha)$

$\dot{x} = -2A\sin(2t + \alpha)$

$\ddot{x} = -4A\cos(2t + \alpha)$

$\therefore\ \ddot{x} = -4x$

$\therefore$ the motion is SH

(ii)

$v^2 = 4A^2\sin^2(2t + \alpha)$

$v^2 = 4A^2(1 - \cos^2(2t + \alpha))$

$v^2 = 4A^2(1-\frac{x^2}{A^2})$

$\therefore\ v^2 = 4(A^2 - x^2)$

(iii)

At x = 5, v = 0

$\therefore\ 0 = 4(A^2 - 25)$

$\therefore\ A^2 = 25$

$\therefore\ A = 5$, since $A > 0$

(iv)

When $t = 0, x = 3, v = 8$

$3 = 5\cos\alpha$ and $8 = -10\sin\alpha$

$\therefore\ \cos\alpha = \frac{3}{5}$ $\quad\therefore\ \sin\alpha = -\frac{4}{5}$

When $t = \frac{\pi}{6}$

$x = 5\cos(\frac{\pi}{3}+\alpha)$

$x = 5(\cos\frac{\pi}{3}\cos\alpha - \sin\frac{\pi}{3}\sin\alpha)$

$x = 5(\frac{1}{2}.\frac{3}{5}+\frac{\sqrt{3}}{2}.\frac{4}{5})$

$\therefore x = \frac{3+4\sqrt{3}}{2}$

Question 6

(a)

$\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}}$

$= \frac{1}{3}\left[\sin^{-1}\frac{3x}{2}\right]_0^{\frac{2}{3}}$

$= \frac{1}{3}\left[\sin^{-1}1 - \sin^{-1}0\right]$

$= \frac{\pi}{6}$

(b)

(i)

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

(ii)

Coeff of $x^4 = 2\times$coeff of x^3

${}^nC_4 = 2\,{}^nC_3$

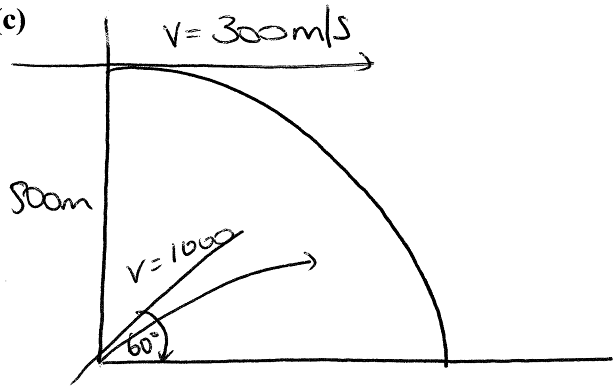
$\frac{n!}{4!(n-4)!} = 2\frac{n!}{3!(n-3)!}$

$\frac{(n-3)!}{(n-4)!} = \frac{2\times 4!}{3!}$

$n - 3 = 8$

$\therefore\ n = 11$

(c)



(i)

$\ddot{x} = 0$

$x = \int 300\,dt$

$x = 300t + c_2$

$\therefore x = 300t$

$\dot{x} = c_1 = 300$

$t = 0, x = 0, \therefore\ c_2 = 0$

(ii)

Initially, $\dot{x} = 1000\cos 60^\circ = 500$

$\ddot{x} = 0$

$x = \int 500\,dt$

$x = 500t + c_3$

$\dot{x} = 500$

$t = 2, x = 0, \therefore\ c_3 = -1000$

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$\therefore x = 500t - 1000$

$300t = 500t - 1000$

$200t = 1000$

$t = 5$

Thus possible interception after 3 seconds

(iii)

After 3 seconds, x and y values for both the missile and rocket must be the same.

Question 7

(a)

(i)

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n \qquad \text{(eqn 1)}$$

Differentiate both side wrt x

$$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2 x + 3\,{}^nC_3 x^2 + \ldots + n\,{}^nC_n x^{n-1}$$

sub $x = 1$

$$n.2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots + n\,{}^nC_n \qquad \text{(eqn 2)}$$

(ii)

$$\sum_{r=0}^{n} (r+1)\,{}^nC_r = 2^{n-1}.(2+n)$$

ie $1.{}^nC_0 + 2.{}^nC_1 + 3.{}^nC_2 + \ldots + (n+1){}^nC_n = 2^{n-1}.(2+n)$

sub x = 1 into eqn 1

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n \qquad \text{(eqn 3)}$$

Add eqn 2 and eqn 3

$2^n + n.2^{n-1} = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n$

Factorising 2^{n-1}

$2^{n-1}(2+n) = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n$

(b)

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(i) $a = \frac{-k}{(x+R)^2}$

$\frac{d}{dx}(\frac{1}{2}v^2) = \frac{-k}{(x+R)^2}$

$\frac{1}{2}v^2 = -k\int(x+R)^{-2}\,dx$

$\frac{1}{2}v^2 = -k\left[\frac{(x+R)^{-1}}{-1}\right] + c$

$\frac{1}{2}v^2 = \frac{k}{(x+R)} + c$

When $x = 0$, $v = U$

$\frac{1}{2}U^2 = \frac{k}{R} + c$

$\therefore c = \frac{1}{2}U^2 - \frac{k}{R}$

$\frac{1}{2}v^2 = \frac{1}{2}U^2 - \frac{k}{R} + \frac{k}{(x+R)}$

$\frac{1}{2}v^2 = \frac{1}{2}U^2 + \frac{kR - k(x+R)}{R(x+R)}$

$\frac{1}{2}v^2 = \frac{1}{2}U^2 - \frac{kx}{R(x+R)}$

$v^2 = U^2 - \frac{2kx}{R(x+R)}$

(ii)

When x = 0 $a = \frac{-k}{R^2}$ = -g $\qquad \therefore k = gR^2$

sub $k = gR^2$ into v^2 from part (i)

$v^2 = U^2 - \frac{2x}{R(x+R)} \times gR^2 \qquad \therefore v^2 = U^2 - \frac{2gRx}{x+R}$

(iii)

If x is small compared with R, so that $\frac{x}{R}$ is close to zero, then $\frac{R}{x+R} \approx 1$

$\therefore v^2 = U^2 - 2gx$

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